Calculus III — Exam Solutions

1. Particle Motion: A particle moves in space with an acceleration given by: $\mathbf{a}(t) = <0, \sin t, \cos t>$. The particle has an initial velocity $\mathbf{v}(0) = <1, -1, 0>$ and an initial position $\mathbf{r}(0) = <0, 0, 2>$

a) Find the velocity vector function $\mathbf{v}(t)$

b) Find the position vector function $\mathbf{r}(t)$

Solution

a) Integrate $\mathbf{a}(t)$:

$$\mathrm{v}(t) = \int \left\langle 0, \sin(t), \cos(t) \right\rangle dt = \left\langle C_1, -\cos(t) + C_2, \sin(t) + C_3 \right\rangle.$$

Apply initial condition $\mathrm{v}(0) = \langle 1, -1, 0 \rangle$:

$$C_1 = 1; -1 + C_2 = -1 \Rightarrow C_2 = 0; 0 + C_3 = 0 \Rightarrow C_3 = 0$$

6 pts - Answer: $\mathrm{v}(t) = \langle 1, -\cos(t), \sin(t) \rangle$

b) Integrate $\mathbf{v}(t)$:

$$\mathbf{r}(t) = \int \left\langle 1, -\cos(t), \sin(t) \right\rangle dt = \left\langle t + K_1, -\sin(t) + K_2, -\cos(t) + K_3 \right\rangle$$

Apply $\mathbf{r}(0) = \langle 0, 0, 2 \rangle$:

$$0 + K_1 = 0 \Rightarrow K_1 = 0; 0 + K_2 = 0 \Rightarrow K_2 = 0; -1 + K_3 = 2 \Rightarrow K_3 = 3$$

6 pts - Answer: $\mathrm{r}(t) = \langle t, -\sin(t), 3 - \cos(t) \rangle$

2. Arc Length and Curvature: Consider the curve defined by the vector function $\mathbf{r}(t) = <0, \cos^3 t, \quad \sin^3 t>$

a) Calculate the length of the path from $t=0$ to $t=\dfrac{\pi}{4}$.

b) Find the unit tangent vector $\mathbf{T}(\mathrm{t})$ and the curvature κ for this curve.

Solution

a) Arc Length

$\mathbf{r}(t) = \left\langle 0, \cos^3(t), \sin^3(t) \right\rangle$

$\mathbf{r}'(t) = \left\langle 0, -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t) \right\rangle$

$|\mathbf{r}'(t)| = \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} = \sqrt{9(\cos^2(t)\sin^2(t))\left(\cos^2(t) + \sin^2(t)\right)} = 3\cos t \sin t$

8 pts - : $L = \displaystyle\int_0^{\frac{\pi}{4}} 3\cos t \sin t \, dt = [\frac{3}{2}\sin^2 t]_0^{\frac{\pi}{4}} = \frac{3}{4}.$

b) Curvature:

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle 0, -\cos(t), \sin(t) \right\rangle$

$\mathbf{T}'(t) = \left\langle 0, \sin(t), \cos(t) \right\rangle \Rightarrow \left|\mathbf{T}'(t) = 1\right|$

6 pts $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1}{3\cos t \sin t}$

3. Limits Evaluate the limit. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ If the limit does not exist, prove that it does not exist.

Solution

One way to analyze this is to consider paths of the form $y = mx^2$

8 pts $\displaystyle\lim_{x\to 0} \frac{2x^2(mx^2)}{x^4+(mx^2)^2} = \lim_{x\to 0} \frac{2mx^4}{\left(1+m^2\right)x^4} = \frac{2m}{\left(1+m^2\right)}$ which clearly depends on m so the limit does not exist.

4. Chain Rule: Let $w = x^2 y + z^2$. The variables x, y, and z are functions of t given by:

$x(t) = \cos(t)$, $y(t) = \sin(t)$, $z(t) = t$. Find $\frac{dw}{dt}$ at $t = \frac{\pi}{3}$. (Use partial derivatives).

Solution 10 pts

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Derivatives:

$w_x = 2xy, \quad w_y = x^2, \quad w_z = 2z$

$x'(t) = -\sin(t), \quad y'(t) = \cos(t), \quad z'(t) = 1$

So at $t = \frac{\pi}{3}$ we have:

$$x = \frac{1}{2}, \quad y = \frac{\sqrt{3}}{2}, \quad z = \frac{\pi}{3}, \quad x' = -\frac{\sqrt{3}}{2}, \quad y' = \frac{1}{2}, \quad z' = 1$$

Substitute values:

$$\frac{dw}{dt} = (2(\frac{1}{2})(\frac{\sqrt{3}}{2}))\left(-\frac{\sqrt{3}}{2}\right) + ((\frac{1}{2})^2(\frac{1}{2}) + (2\frac{\pi}{3})(1) = -\frac{3}{4} + \frac{1}{8} + \frac{2\pi}{3} = -\frac{5}{8} + \frac{2\pi}{3}$$

5. Tangent Planes and Differentials: Consider the surface $z = x^2 + y^2 - 2x - 4y$.

a) Find the equation of the tangent plane to the surface at the point (2, -1, 5).

b) Use the differential dz to estimate the value of z at the point (2.01, -0.98).

Solution

a) Let $F(x, y, z) = x^2 + y^2 - 2x - 4y - z = 0$. Then $\nabla F = <2x - 2, 2y - 4, -1>$

At (2,-1,5) : $\nabla F = \langle 4 - 2, -2 - 4, -1 \rangle = \langle 2, -6, -1 \rangle$.

6 pts - Plane equation: $2(x - 2) - 6(y + 1) - 1(z - 5) = 0$ or $2x - 6y - z = 5$

b) $dz = (2x - 2)\,dx + (2y - 4)\,dy$.

At (2,-1): $dz = 2dx - 6dy = 2(0.01) - 6(0.02) = -0.1$.

Since $dx = 2.01 - 2$ and $dy = -0.98 - (-1) = 0.02$

So $dz = 2(0.01) - 6(0.02) = 0.02 - 0.12 = -0.1$

4 pts - $z \approx z_{\text{old}} + dz = 5 - 0.1 = 4.9$

6. Directional Derivatives: Let $f(x,y)=xe^{xy}$.

a) Find the gradient vector ∇f at the point P(1, 0).

b) Calculate the directional derivative of f at P(1, 0) in the direction of $\mathbf{v}=\langle 3,-4\rangle$.

Solution

a) $\nabla f=\langle f_x, f_y\rangle=\langle e^{xy}(1+xy), x^2e^{xy}\rangle$

6 pts - At (1,0) $\nabla f=\langle 1,1\rangle$

b) Vector $\mathbf{v}=\langle 3,-4\rangle$. $|\mathbf{v}|=\sqrt{9+16}=5;$ Unit vector $\mathbf{u}=\left\langle \frac{3}{5},-\frac{4}{5}\right\rangle$.

4 pts - $D_{\mathbf{u}}f=\nabla f\cdot\mathbf{u}=\left\langle 1,1\right\rangle\cdot\left\langle\frac{3}{5},-\frac{4}{5}\right\rangle=\frac{3}{5}-\frac{4}{5}=-\frac{1}{5}$

7. Extreme Values: Find all local maximums, local minimums, and saddle points for the function: $f(x,y)=x^3-3x+3y^2-6y$ and classify each critical point.

Solution 12 pts

$f_x=3x^2-3=0\Rightarrow x^2=1\Rightarrow x=\pm 1;\ f_y=6y-6=0\Rightarrow y=1$

Critical points: (1, 1) and (-1, 1).

Second Derivatives: $f_{xx}=6x,\quad f_{yy}=6,\quad f_{xy}=0$

Determinant $D=f_{xx}f_{yy}-(f_{xy})^2=36x$

At (1, 1): $D=36(1)=36>0$ and $f_{xx}=6>0$. **Local Minimum at (1,1) with $f=-5$.**

At (-1, 1): $D=36(-1)=-36<0$. **Saddle Point at (-1, 1) with $f=-1$.**

8. Lagrange Multipliers: Use the method of Lagrange Multipliers to find the maximum and minimum values of the function $f(x,y)=x+2y$ subject to the constraint $x^2+y^2=5$.

Solution 12 pts

$\nabla f=\langle 1,2\rangle, \nabla g=\langle 2x,2y\rangle$

$\langle 1,2\rangle=\lambda\langle 2x,2y\rangle\Rightarrow 1=2\lambda x\Rightarrow x=\frac{1}{2\lambda}$ and $2=2\lambda y\Rightarrow y=\frac{1}{\lambda}$. So $y=2x$

Constraint: $x^2+(2x)^2=5\Rightarrow 5x^2=5\Rightarrow x=\pm 1$.

If $x=1, y=2\Rightarrow f=1+2(2)=5$ (Maximum).

If $x=-1, y=-2\Rightarrow f=-1+2(-2)=-5$ (Minimum).

9. Taylor's Formula: Find the second-degree Taylor polynomial $P_2(x, y)$ for the function $f(x, y) = \cos(x)\sin(y)$ centered at the point $(0, \frac{\pi}{2})$

Solution

At $(0, \frac{\pi}{2})\cdot$

$f = \cos x \sin y$ Value: $1 \cdot 1 = 1$

$f_x = -\sin x \sin y$ Value: $0 \cdot 1 = 0$

$f_y = \cos x \cos y$ Value: $1 \cdot 0 = 0$

$f_{xx} = -\cos x \sin y$ Value: $-1 \cdot 1 = -1$

$f_{yy} = -\cos x \sin y$ Value: $1 \cdot (-1) = -1$

$f_{xy} = -\sin x \cos y\cdot$ Value: $0 \cdot 0 = 0.$

Polynomial:

$$P_2(x,y) = 1 + 0(x) + 0(y - \frac{\pi}{2}) + \frac{1}{2}[-1(x-0)^2 + 2(0)(x)(y - \frac{\pi}{2}) + (-1)(y - \frac{\pi}{2})^2]$$

8 pts - $P_2(x, y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}(y - \frac{\pi}{2})^2$

10. Double Integral: Evaluate the double integral over the rectangular region $R = [0, 1] \times [0, 2]$ (where): $\iint_R (4x + 6y^2)\, dA$

Solution

$$\int_0^2 \int_0^1 (4x + 6y^2)\, dx\, dy$$

Inner Integral : $[2x^2 + 6y^2x]_0^1 = (2 + 6y^2) - 0 = 2 + 6y^2\cdot$

Outer Integral $\int_0^2 (2 + 6y^2)\, dy = [2y + 2y^3]_0^2\cdot$

Evaluate: $(2(2) + 2(8)) - 0 = 4 + 16 = 20$

8 pts - Answer: 20